A Method to construct the Sparse-paving Matroids over a Finite Set

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Abstract

We give a method to construct the sparse-paving matroids over a finite set S. With it, we give an injective set-function $\Psi_r : \mathrm{Matroid}_{n,r} \to \sqcup_{j=1}^{2\binom{r}{[r/2]}} \mathrm{Sparse}_{n,r}$ where $(\mathrm{Sparse}_{n,r})$ $\mathrm{Matroid}_{n,r}$ is the set of all (sparse-paving) matroids of rank r, over a set S of cardinality n. Then, we give another proof of $\lim_{n\to\infty} \frac{\log_2 |\mathrm{Matroid}_{n,r}|}{\log_2 |\mathrm{Sparse}_{n,r}|} = 1$ and some new bounds of the cardinalities of these sets.

Keywords: matroid, paving matroid, sparse-paving matroid, combinatorial geometries, lattice of a matroid.

Introduction

We recall that a **matroid** $M = (S, \mathcal{I})$ consists of a finite set S and a collection $\mathcal{I}$ of subsets of S (called the **independent sets** of M) satisfying the following **independence axioms**:

($\mathcal{I}$1) The empty set $\emptyset \in \mathcal{I}$.
($\mathcal{I}$2) If $X \in \mathcal{I}$ and $Y \subseteq X$ then $Y \in \mathcal{I}$.
($\mathcal{I}$3) Let $U, V \in \mathcal{I}$ with $|U| = |V| + 1$ then $\exists x \in U \setminus V$ such that $V \cup \{x\} \in \mathcal{I}$.

A subset of S which does not belong to $\mathcal{I}$ is called a *dependent set* of M.

A **basis** [respectively, a **circuit**] of M is a maximal independent [resp. minimal dependent] set of M. The **rank** of a subset $X \subseteq S$ is $\mathrm{rk}X := \max\{|A| ; A \subseteq X \text{ and } A \in \mathcal{I}\}$ and the rank of the matroid M is $\mathrm{rk}M := \mathrm{rk}S$. A **closed** subset (or **flat**) of M is a subset $X \subseteq S$ such that for all $x \in S \setminus X$, $\mathrm{rk}(X \cup \{x\}) = \mathrm{rk}X + 1$. Then can be defined the **closure operator** $cl : \mathcal{P}S \to \mathcal{P}S$ on the *power set* of S, as follows: $cl(X) := \min\{Y \subseteq S; X \subseteq Y \text{ and } Y \text{ is closed in } M\}$. The **lattice of** a matroid M, denoted by $\mathcal{L}_M$ is the lattice defined by the closed sets of M, ordered by inclusion where the meet is the intersection and the join the closure of the union of sets. For general references of Theory of Matroids, see [15], [11], [14] and [12]. For references of theory of lattices and theory of lattices of matroids, see [4], [6].

A matroid is **paving** if it has no circuits of cardinality less than rk$M$. And a matroid M is **sparse-paving** if M and its dual M^* are paving matroids.

In [3], Blackburn, Crapo and Higgs wrote: "In the enumeration of (non-isomorphic) matroids on a set of 9 or less elements, (sparse-)paving matroids predominate. Does this hold in general?". There are several results which suggest that the answer should be positive, see for example [2], [1], [5]. We give a method to construct the sparse-paving matroids over a finite set S and an injective set-function $\Psi_r : \mathrm{Matroid}_{n,r} \to \sqcup_{j=1}^{2\binom{r}{[r/2]}} \mathrm{Sparse}_{n,r}$ where $(\mathrm{Sparse}_{n,r})$ $\mathrm{Matroid}_{n,r}$ is the set of all (sparse-paving) matroids of rank r, over a set S of cardinality n. Then, we give another proof of $\lim_{n\to\infty} \frac{\log_2 |\mathrm{Matroid}_{n,r}|}{\log_2 |\mathrm{Sparse}_{n,r}|} = 1$ and some new bounds of the cardinalities of these sets.

The material is organized as follows: In section1, we give more definitions and known results that are useful along the paper. Also, we give an abstract construction of the sparse-paving matroids, M, which leads us to a method to construct them explicitly, using matrices of $r-$subsets of S, for any r. In section2, we prve the following inequalities: $\left|\mathrm{Sparse}_{n,r}\right| \leq \left|\mathrm{Sparse}_{n+1,r}\right| \leq \left|\mathrm{Sparse}_{n,r}\right| + \left|\mathrm{Sparse}_{n,r-1}\right|$ where $\mathrm{Sparse}_{n,r} := \{M = (S, \mathcal{I}); M \text{ is a sparse-paving matroid of } \mathrm{rk}M = r\}$ and $|S| = n$.

In section3, we give a construction of a partition of the $r-$subsets of S, $\binom{S}{r} = \sqcup_{i=1}^{\gamma} \mathcal{U}_i$ such that each $\mathcal{U}_i$ define a sparse-paving matroid of rank r and $\gamma = 2\binom{r}{[r/2]}$. In section4, we give an injective function that maps each matroid over S $\mathrm{Matroid}_{n,r}$, to a disjoint union of $2\binom{r}{[r/2]}$ sets of sparse-paving matroids on S of rank r. And new bounds of these cardinalities are given.

1 A description of the Sparse-paving Matroids through their set of circuits.

The study of sparse-paving and paving matroids helps to understand the behavior of the matroids in general and important examples of matroids are indeed sparse-paving matroids, as the combinatorial finite geometries. In 1959, Hartmanis [6] introduced the definition of paving matroid through the concept of d-partition in number theory. Then, following a Rota's suggestion, Welsh [14, (1976)] called these matroids paving. Later, Oxley [12] generalized the definition of paving matroid to include all possible ranks. And Jerrum [7] introduced

the notion of sparse-paving matroids. It is known that the sparse-paving matroids of rank ≥ 2 have lattices which are atomic, semimodular and satisfies the Jordan-Hölder condition.

1.1 More definitions, notations and known results.

Let S be a set of n elements.

a. Any matroid $M = (S, \mathcal{I})$ is completely determined by its set of basis, $\mathcal{B}$. Namely, $\mathcal{I} = \{X \subseteq S; \exists B \in \mathcal{B} \text{ with } X \subseteq B\}$.

b. Let $M = (S, \mathcal{I})$ be a matroid of rank $\text{rk}M$. Any circuit X of M, has cardinality $|X| \leq \text{rk}M + 1$.

c. Let $M = (S, \mathcal{I})$ be a matroid of rank $\text{rk}M$. Denote by $M^* = (S, \mathcal{I}^*)$ the **dual matroid** of M whose set of basis is $\mathcal{B}^* := S \backslash \mathcal{B}$.

A matroid M is called a **sparse-paving matroid** if M and its dual M^* are paving matroids.

d. Examples of sparse-paving matroids:

d.1. A matroid is called **uniform** of rank r over a set of n elements, denoted by $U_{r,n}$, if all the $r-$subsets are basis. The dual matroid $U_{r,n}^*$ of a uniform matroid is again uniform with rank $n - r$. Then any uniform matroid is sparse-paving. (This occurs, for example, when $r = 0$ or $r = n$).

d.2. Any matroid of rank 1 is paving (since the empty set is always independent).

d.3. By (d.1) and (d.2), for $n = 1, 2$ any matroid on S is sparse-paving.

e. If a matroid M has rank $\text{rk}M \geq 2$, the lattice of M, $\mathcal{L}_M$ is atomic (i.e., all the subsets of rank 1 are closed), semimodular and satisfies the Jordan-Hölder condition. An important class of these matroids are the combinatorial finite geometries, [12].

1.2. By (1.1.d), along this paper we can assume that $n \geq 3$ and $r \geq 2$.

Let S be a set of cardinality $|S| = n$. Denote by $\binom{S}{t} := \{X \subseteq S \, ; \, |X| = t\}$ for $0 \leq t \leq n$ the subsets of S of cardinality t (called $t-$**subsets**). Let $M = (S, \mathcal{I})$ be a **paving matroid of rank** $\text{rk}M$. Denote by $\mathcal{B}$ [resp. $\mathcal{C}_{\text{rk}M}$] the set of the basis [resp. $\text{rk}M$-circuits] of M.

Lemma [8][5]. *For $n \geq 3$ and $\text{rk}M \geq 2$, there is an equivalent definition of being a sparse-paving matroid. Namely, A matroid $M = (S, I)$ with $|S| \geq 3$ and $\text{rk}M \geq 2$ is a sparse-paving matroid if and only if its set of $\text{rk}M$-circuits, $C_{\text{rk}M}$ satisfies the following property:*

$$(**) \; \textit{For all } X, Y \in C_{\text{rk}M} \textit{ we have } |X \cap Y| \leq rkM - 2$$

1.3. The next result is the counterpart of lemma in (1.2). That is, let S be a set of cardinality $n \geq 3$ and $2 \leq r \leq n - 1$. Then *any* set $\mathcal{C} \subseteq \binom{S}{r}$ of r-subsets of S satisfying property $(**)$ defines a sparse-paving matroid of rank r with $\mathcal{C}$

as its set of r-circuits. In other words, in this case, the ordered pair $(S, \mathcal{I})$ with $\mathcal{I} := \{X \subseteq S; \exists B \in \binom{S}{r} \backslash \mathcal{C}\}$ is in fact a matroid (ie., $\mathcal{I}$ satisfies the independent axioms of a matroid, see(Introduction)) and by (1.2), $(S, \mathcal{I})$ is sparse-paving.

Proposition. *Let S be a set of cardinality $|S| = n \geq 3$ and $2 \leq r \leq n - 1$. Let $C \subset \binom{S}{r}$ be a set of $r-$subsets of S, satisfying the following property*

$$(**): \ \forall X, Y \in C \ \text{with} \ X \neq Y \ \text{then} \ |X \cap Y| \leq r - 2.$$

Define $M := (S, I)$ where $B := \binom{S}{r} \backslash C$ and $I := \{X \subseteq S; \exists B \in B \ \text{with} \ X \subseteq B\}$. Then, (A). M is a matroid of $\mathrm{rk} M = r$ and (B). M is sparse-paving.

Proof. Let S be a set and take a subset $\mathcal{C} \subset \binom{S}{r}$ satisfying the property $(**)$. Take $M = (S, \mathcal{I})$ with set of basis $\mathcal{B} = \binom{S}{r} \backslash \mathcal{C}$.

A. *To prove M is a matroid of rank $\mathrm{rk} M = r$.*

For this proof, we will use an equivalent definition of matroid, which says:

Let $M = (S, \mathcal{I})$ is a matroid if and only if $\mathcal{I}$ satisfies $(\mathcal{I}1),(\mathcal{I}2)$ as in the introduction and $(\mathcal{I}3)'$: let $B_1, B_2 \in \mathcal{B}$ be two basis of M and $x \in B_1 \backslash B_2$. To prove $\exists y \in B_2 \backslash B_1$ such that $(B_1 \backslash \{x\}) \cup \{y\} \in \mathcal{B}$.

case a. If $|S| = 3$ and $\mathrm{rk} M = 2$, the possibilities for $\mathcal{C}$ to have property $(**)$ are $\mathcal{C} = \emptyset$ or $|\mathcal{C}| = 1$. In both cases, M is matroid and it is sparse-paving.

case b. $|S| \geq 4$.

$(\mathcal{I}1)$ To prove that $\emptyset$ is an independent set. It is enough to prove that $\mathcal{B}$ is not empty.

Since $n \geq 4$, $2 \leq r \leq n - 1$ and $S = \{1, ..., r, r + 1, ..., n\}$. Take $A_1 = \{1, ..., r - 1, r\}$, $A_2 = \{1, ..., r - 1, r + 1\}$ which are subsets of S with cardinality r and $|A_1 \cap A_2| = r - 1$. Then by $(**)$, $\exists i \in \{1, 2\}$ such that $A_i \in \mathcal{B}$. Then $\mathcal{B} \neq \emptyset$.

$(\mathcal{I}2)$ Let $Y \subseteq X \subseteq S$ such that $\exists B \in \mathcal{B}$ with $X \subseteq B$. Then $Y \subseteq B$, that is Y is independent, by definition.

$(\mathcal{I}3)'$ Now, let $B_1, B_2 \in \mathcal{B}$ be two basis of M and $x \in B_1 \backslash B_2$. To prove $\exists y \in B_2 \backslash B_1$ such that $(B_1 \backslash \{x\}) \cup \{y\} \in \mathcal{B}$.

$\mathcal{I}3'.1.$ Assume $m := |B_2 \backslash B_1| = 1$. That is, $B_2 \cap B_1 = B_1 \backslash \{x\}$ and $B_2 = (B_1 \backslash \{x\}) \cup \{y\}$ for some $y \in S$. Then $(B_1 \backslash \{x\}) \cup \{y\} \in \mathcal{B}$.

$\mathcal{I}3'.2.$ Let define $m := |B_2 \backslash B_1| \geq 2$ and let $B_2 = (B_1 \cap B_2) \cup \{y_1, y_2, y_3, ..., y_m\}$. Define $A_i := (B_1 \backslash \{x\}) \cup \{y_i\}$ for $i = 1, ..., m$. Since $\forall i \neq j$, $|A_i \cap A_j| = r - 1$ and $m \geq 2$, by $(**)$, $\exists A_{i_0} \in \mathcal{B}$. Therefore, $(B_1 \backslash \{x\}) \cup \{y_i\} = A_{i_0} \in \mathcal{B}$, and M is a matroid.

Rank: By definition of M, $\mathrm{rk} M = r$.

B. *To prove M is a sparse-paving matroid.*

B.1. First we will prove that M is a paving matroid. Equivalently, to prove $\forall Z \subseteq S$ of $|Z| = \mathrm{rk} M - 1$, $Z \in \mathcal{I}$. This proof is similar to the one of $(\mathcal{I}1)$. Namely:

Let $\mathrm{rk} M \leq n - 1$. Since $n \geq 3$ and $|Z| = \mathrm{rk} M - 1$, we have $S = Z \cup \{x_1, x_2, ..., x_m\}$ with $m \geq 2$. Let denote $A_i := Z \cup \{x_i\}$ for $i = 1, 2, ..., m$. By $(**)$ and $m \geq 2$, $\exists i_0 \in \{1, ..., m\}$ such that $(Z \subset) A_{i_0} \in \mathcal{B}$. Then $Z \in \mathcal{I}$.

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B.2. And by (1.2), M is a sparse-paving matroid.$\blacksquare$

2 Recursive construction of bounds for the cardinality of the sparse-paving matroids on a finite set S with $|S| = n$.

Recall, $\text{Sparse}_{n,r} := \{M = (S,\mathcal{I}); M \text{ is a sparse-paving matroid with } \text{rk}M = r\}$ where $|S| = n$. By (1.3), to construct a sparse-paving matroid on S of rank $r \geq 2$, it is enough to have a set $\mathcal{U} \subset \binom{S}{r}$ satisfying property $(**)$: $\forall X, Y \in \mathcal{U}$ with $X \neq Y$, $|X \cap Y| \leq r - 2$, using this, we prove $\left|\text{Sparse}_{n,r}\right| \leq \left|\text{Sparse}_{n+1,r}\right| \leq \left|\text{Sparse}_{n,r}\right| + \left|\text{Sparse}_{n,r-1}\right|$.

In other hand, we prove that we can construct that $\mathcal{U} \subset \binom{S}{r}$ satisfying property $(**)$ and $\frac{1}{r(n-r)+1}\left|\binom{S}{r}\right| \leq |\mathcal{U}|$, this implies, $\left|\text{Sparse}_{n,r}\right| \geq 2^{\left[\frac{1}{r(n-r)+1}\left|\binom{S}{r}\right|\right]}$.

Also we prove that any $\mathcal{U} \subset \binom{S}{r}$ with property $(**)$, $|\mathcal{U}| \leq \frac{1}{n-r}\left|\binom{S}{r+1}\right|$.

2.1. Lemma. $\text{Sparse}_{n,r} \hookrightarrow \text{Sparse}_{n+1,r} \hookrightarrow \text{Sparse}_{n,r} \sqcup \text{Sparse}_{n,r-1}$.

Therefore, $\left|\text{Sparse}_{n,r}\right| \leq \left|\text{Sparse}_{n+1,r}\right| \leq \left|\text{Sparse}_{n,r}\right| + \left|\text{Sparse}_{n,r-1}\right|$.

Proof: Denote by $S := \{1, 2, ..., n\}$ and $\widehat{S} := S \cup \{n+1\}$.

2.1.1. The first inequality is given by the following canonical injective set-function $\iota_n : \text{Sparse}_{n,r} \hookrightarrow \text{Sparse}_{n+1,r}$: Let $M = (S, \mathcal{I})$ be a sparse-paving matroid of $\text{rk}M = r$ and $\mathcal{C}_r$ its set of $r-$circuits (recall that in the sparse-paving case $\binom{S}{r} = \mathcal{B} \sqcup \mathcal{C}_r$).

Define $\iota_n(M) =: (\widehat{S}, \widehat{\mathcal{I}})$ where the $(\widehat{S}, \widehat{\mathcal{I}})$'s set of basis is $\widehat{\mathcal{B}} := \mathcal{B} \cup \{X \in \binom{\widehat{S}}{r}; n+1 \in X\}$ and $\widehat{\mathcal{I}} = \{Y \subseteq \widehat{S}; \exists \widehat{B} \in \widehat{\mathcal{B}} \text{ with } Y \subseteq \widehat{B}\}$. Then $(\widehat{S}, \widehat{\mathcal{I}})$'s $r-$circuit set $\widehat{\mathcal{C}}_r = \mathcal{C}_r$, the set of $r-$circuits of M. By (1.2), $\mathcal{C}_r(= \widehat{\mathcal{C}}_r)$ has property $(**)$ and by (1.3), $(\widehat{S}, \widehat{\mathcal{I}})$ is a sparse-paving matroid. And ι_n is injective, by definition of $\widehat{\mathcal{B}}$.

2.1.2. To prove the second inequality define
$\zeta_{n+1} : \text{Sparse}_{n+1,r} \hookrightarrow \text{Sparse}_{n,r} \sqcup \text{Sparse}_{n,r-1}$ an injective set-function as follows: let $\widehat{M} = (\widehat{S}, \widehat{\mathcal{I}})$ be a sparse-paving matroid of $\text{rk}\widehat{M} = r$ and $\widehat{\mathcal{C}}_r$ its set of $r-$circuits (recall, $\binom{\widehat{S}}{r} = \widehat{\mathcal{B}} \sqcup \widehat{\mathcal{C}}_r$). Define $\zeta_{n+1}(\widehat{M}) := (M_1, r) \cup (M_2, r-1)$ as follows: $\widehat{\mathcal{C}}_r = \{X \in \widehat{\mathcal{C}}_r; n+1 \notin X\} \sqcup \{X \in \widehat{\mathcal{C}}_r; n+1 \in X\}$ a disjoint union of $\mathcal{C}_r^{(1)} := \{X \in \widehat{\mathcal{C}}_r; n+1 \notin X\}$ and $\{X \in \widehat{\mathcal{C}}_r; n+1 \in X\}$.

Take $M_1 = (S, \mathcal{B}_1)$ with $\mathcal{B}_1 = \binom{S}{r} \backslash \mathcal{C}_r^{(1)}$. Now, since $\widehat{\mathcal{C}}_r$ satisfies property $(**)$ for r, see (1.3), we get that $M_1 \in \text{Sparse}_{n,r}$.

In other hand, define $\mathcal{C}_r^{(2)} := \{X \backslash \{n+1\}; \ X \in \widehat{\mathcal{C}}_r \text{ and } n+1 \in X\} \subset \binom{S}{r-1}$ and observe again, since $\widehat{\mathcal{C}}_r$ satisfies property $(**)$ for r. Now by property $(**)$ for r, if $X, Y \in \widehat{\mathcal{C}}_r$ with $n+1 \in X \cap Y$ then $|(X \backslash \{n+1\}) \cap (Y \backslash \{n+1\})| = |X \cap Y| - 1 \le (r-2) - 1 = (r-1) - 2$. In other words by (1.3), $\mathcal{C}_{r-1}^{(2)}$ defines a sparse-paving matroid on S of rank $r-1$, with $\mathcal{C}_{r-1}^{(2)}$ as $r-1$–circuits of the matroid denoted as M_2.

Then ζ_{n+1} is a well defined set-function. And it is injective by (1.1.a) and $\widehat{\mathcal{C}}_r = \mathcal{C}_r^{(1)} \cup \{Z \cup \{n+1\}; \ Z \in \mathcal{C}_{r-1}^{(2)}\}$.

2.1.3. Therefore, $\left|\text{Sparse}_{n,r}\right| \le \left|\text{Sparse}_{n+1,r}\right| \le \left|\text{Sparse}_{n,r}\right| + \left|\text{Sparse}_{n,r-1}\right|. \blacksquare$

2.2. A bound for $|\mathcal{C}_{\mathrm{rk}M}|$ the cardinality of the set of $\mathrm{rk}M$-circuits, $\mathcal{C}_{\mathrm{rk}M}$, of a sparse-paving matroid M.

Since we want to find relations between the cardinality of all matroids on a finite n–set S of rank r, $|\text{Matroid}_{n,r}|$ and its subset of sparse-paving matroids, $\left|\text{Sparse}_{n,r}\right|$, in this section we give easy-finding bounds of the set of r–circuits of the sparse-paving matroids (see, property $(**)$ in (1.2)).

Recall, the following property on $\mathcal{U}$ and r, $(**)$: For all $X, Y \in \mathcal{U}$ we have $|X \cap Y| \le r - 2$.

Technical steps to construct sets with property $(**)$**:** Let S with $|S| = n$ and $2 \le r \le n-1$. Let $X_1 \in \binom{S}{r}$ be fix.

A. *Want to find* $\{X \in \binom{S}{r}; \ |X \cap X_1| \le r-2\}$. *Equivalent, find* $\{A \in \binom{S}{r}; \ |A \cap X_1| = r-1\}$.

A.1. Take all the $r+1$–subsets of S containing X_1. Namely, $\{Y_{1,1}, Y_{1,2}, ..., Y_{1,n-r}\} = \{Y \in \binom{S}{r+1}; \ X_1 \subset Y\}$. That is, $Y_{1,i} = X_1 \cup \{v_i\}$ where $S \backslash X_1 = \{v_1, ..., v_{n-r}\}$.

A.2. By construction, $\{A \in \binom{S}{r}; \ |X_1 \cap A| = r-1\} = \{A \in \binom{S}{r}; \ \exists i = 1, ..., n-r \text{ such that } A \subset Y_{1,i}\} \backslash \{X_1\}$. And then $\left|\{A \in \binom{S}{r}; \ |X_1 \cap A| = r - 1\} \cup \{X_1\}\right| = r(n-r) + 1$.

A.3. And $\{A \in \binom{S}{r}; \ |X_1 \cap A| \le r - 2\} = \binom{S}{r} \backslash \{A \in \binom{S}{r}; \ |X_1 \cap A| = r - 1\}$ has cardinality $\binom{n}{r} - r(n-r) - 1$.

B. *Next choose and fix* $X_2 \in \{A \in \binom{S}{r}; \ |X_1 \cap A| \le r - 2\}$. *To get a set of* r–*subsets of* S *with property* $(**)$*, we have to make stepA for* X_2*:*

B.1. By (A.1) for X_2, let $\{Y_{2,1}, Y_{2,2}, ..., Y_{2,n-r}\} = \{Y \in \binom{S}{r+1}; \ X_2 \subset Y\}$.

Thus, $\binom{S}{r} \backslash \{A \in \binom{S}{r}; \ \exists h = 1, 2, \ \exists i = 1, ..., n - r \text{ such that } A \subset Y_{h,i}\} = \{A \in \binom{S}{r}; \ |X_1 \cap A| \le r - 2 \text{ and } |X_2 \cap A| \le r - 2 \}$. Also by construction, $|X_1 \cap X_2| \le r - 2$.

C. *By construction,* $\{Y_{1,1}, Y_{1,2}, ..., Y_{1,r-n}\} \cap \{Y_{2,1}, Y_{2,2}, ..., Y_{2,r-n}\} = \emptyset$, *since* $X_1 \nsubseteq Y_{2,j}$ *and* $X_2 \nsubseteq Y_{1,j}$ *for all* $j = 1, ..., r - n$.

D. *In this way, by (C), we can continue the procedure at most* $\left[\frac{1}{n-r}\binom{n}{r+1}\right]$ *steps. That is, we can construct* $U \subset \binom{S}{r}$ *satisfying property* $(**)$ *with* $|\mathcal{U}| \leq \left[\frac{1}{n-r}\binom{n}{r+1}\right]$.

E. And all the sets $\mathcal{C} \subseteq \binom{S}{r}$ satisfying property $(**)$ is a subset of a $\mathcal{U}$ that can be built with this procedure.

We proved the following

Lemma. *Let S be a set of cardinality n and let $2 \leq r \leq n-1$.*

a) Assume that $U \subset \binom{S}{r}$ satisfies property $(**)$. Then $|\mathcal{U}| \leq \frac{1}{n-r}\binom{n}{r+1}$.

b) There exists $U \subset \binom{S}{r}$ satisfying property $(**)$ with cardinality at least $\frac{1}{r(n-r)+1}\binom{n}{r} \leq |\mathcal{U}|$.

2.3. For the next result, see also [10, (4.8)].

Corollary. *Let S be a set of cardinality n and let $2 \leq r \leq n-1$. Let $M = (S, I)$ be a sparse-paving matroid of rank r and C_r its set of $r-$circuits. Then $|\mathcal{C}_r| \leq \frac{1}{n-r}\binom{n}{r+1}$.*

2.4. Corollary. *Let S be a set of cardinality n and let $2 \leq r \leq n-1$. Then $\left|\text{Sparse}_{n,r}\right| \geq 2^{\left[\frac{1}{r(n-r)+1}\binom{n}{r}\right]}$.*

Proof. By lemma(b) in (2.2), there exists $\mathcal{U} \subset \binom{S}{r}$ with property $(**)$ and $\frac{1}{r(n-r)+1}\binom{n}{r} \leq |\mathcal{U}|$. Let $\mathcal{P}(\mathcal{U}) = \{X \subseteq \mathcal{U}\}$ be the power set of $\mathcal{U}$. Then $\forall \mathcal{C} \in \mathcal{P}(\mathcal{U})$, $\mathcal{C}$ satisfies property $(**)$, then $\mathcal{C}$ defines a sparse-paving matroid. Moreover, if $\mathcal{C} \neq \mathcal{C}'$ in $\mathcal{P}(\mathcal{U})$, their respective sparse-paving matroids are different. And since $\frac{1}{r(n-r)+1}\binom{n}{r} \leq |\mathcal{U}|$, we have $2^{\left[\frac{1}{r(n-r)+1}\binom{n}{r}\right]} \leq |\mathcal{P}(\mathcal{U})| = 2^{|\mathcal{U}|} \leq \left|\text{Sparse}_{n,r}\right|$. $\blacksquare$

3 A method to construct sets $\mathcal{U}$ with property $(**)$: $\forall X, Y \in \mathcal{U}$ with $X \neq Y$, $|X \cap Y| \leq r - 2$.

In this section, we will construct matrices of $r-$subsets of S from a fixed $r-$subset X having the following properties:

a) Any $r-$subset of S is an entry of exactly one of these matrices.

b) In each matrix, any two entries which are in different rows and different columns have intersection less or equal to $r - 2$.

c) Any two entries in different matrices, S_h and $S_{h'}$ with $|h - h'| \geq 2$, have intersection less or equal to $r - 2$.

3.1. Let S be a set of cardinality n, $2 \leq r \leq n-1$. Fix $X \in \binom{S}{r}$. For each $0 \leq h \leq r$, let $\binom{X}{h} = \{A_1^{(h)}, ..., A_{\binom{r}{h}}^{(h)}\}$ be the $h-$subsets of X and $\binom{S \backslash X}{r-h} = \{Z_1^{(h)}, ..., Z_{\binom{n-r}{r-h}}^{(h)}\}$ be the $(r-h)-$subsets of $S \backslash X$.

For each $0 \leq h \leq r$ and $n-r \geq r-h$, we build the following $\binom{|S \backslash X|}{r-h} \times \binom{|X|}{h}-$matrix, s_h:

$$s_h := \begin{bmatrix} A_1^{(h)} \cup Z_1^{(h)} & \cdots & A_{\binom{r}{h}}^{(h)} \cup Z_1^{(h)} \\ A_1^{(h)} \cup Z_2^{(h)} & \cdots & A_{\binom{r}{h}}^{(h)} \cup Z_2^{(h)} \\ \vdots & \vdots & \vdots \\ A_1^{(h)} \cup Z_{\binom{n-r}{r-h}}^{(h)} & \cdots & A_{\binom{r}{h}}^{(h)} \cup Z_{\binom{n-r}{r-h}}^{(h)} \end{bmatrix}_{\binom{S \backslash X}{r-h} \times \binom{X}{h}}$$

3.1.1. Properties of s_h:

a) *By construction of the matrices S_h. $\forall Y \in \binom{S}{r}$, there exists a unique $0 \leq h \leq r$ such that Y is an entry of S_h.*

b) *Let $0 \leq h \leq r$ and s_h. Now take $1 \leq i \neq j \leq \binom{n-r}{r-h}$, $1 \leq t \neq k \leq \binom{r}{h}$ and the entries $A_t^{(h)} \cup Z_i^{(h)}$, $A_k^{(h)} \cup Z_j^{(h)}$. Then $\left| \left(A_t^{(h)} \cup Z_i^{(h)} \right) \cap \left(A_k^{(h)} \cup Z_j^{(h)} \right) \right| \leq r - 2$. That is, a pair of entries in different columns and different rows have intersection $\leq r - 2$.*

c) *For $0 \leq h, h' \leq r$ such that $|h - h'| \geq 2$ and for all i, j, t and k, $\left| \left(A_t^{(h)} \cup Z_i^{(h)} \right) \cap \left(A_k^{(h')} \cup Z_j^{(h')} \right) \right| \leq r - 2$.*

Proof. (b). $\left| \left(A_t^{(h)} \cup Z_i^{(h)} \right) \cap \left(A_k^{(h)} \cup Z_j^{(h)} \right) \right| = \left| A_t^{(h)} \cap A_k^{(h)} \right| + \left| Z_i^{(h)} \cap Z_j^{(h)} \right| \leq (h-1) + (r - h - 1) = r - 2$, since $A_t^{(h)} \neq A_t^{(h)} \subseteq X$ and $Z_i^{(h)} \neq Z_j^{(h)} \subseteq S \backslash X$.

(c) Let $0 \leq h, h' \leq r$ and $|h - h'| \geq 2$. Take any $1 \leq i, j \leq \binom{n-r}{r-h}$ and $1 \leq t, k \leq \binom{r}{h}$. To prove that $\left| \left(A_t^{(h)} \cup Z_i^{(h)} \right) \cap \left(A_k^{(h')} \cup Z_j^{(h')} \right) \right| \leq r - 2$.

Since $|h - h'| \geq 2$, we can assume that $h = h' + m$ with $2 \leq m \leq r - h'$. Then $\left| \left(A_t^{(h)} \cup Z_i^{(h)} \right) \cap \left(A_k^{(h')} \cup Z_j^{(h')} \right) \right| = \left| A_t^{(h)} \cap A_k^{(h')} \right| + \left| Z_i^{(h)} \cap Z_j^{(h')} \right| \leq h + (r - h') = r - m \leq r - 2.\blacksquare$

3.2. Example. Let $S = \{1, 2, 3, 4, 5, 6\}$, $r = 3$ and fix $X = \{1, 2, 3\}$.

$$s_0 := \begin{bmatrix} \{4, 5, 6\} \end{bmatrix}_{\binom{S \backslash X}{3} \times \binom{X}{0}},$$

$$s_1 := \begin{bmatrix} \{1\} \cup \{4, 5\} & \{2\} \cup \{4, 5\} & \{3\} \cup \{4, 5\} \\ \{1\} \cup \{4, 6\} & \{2\} \cup \{4, 6\} & \{3\} \cup \{4, 6\} \\ \{1\} \cup \{5, 6\} & \{2\} \cup \{5, 6\} & \{3\} \cup \{5, 6\} \end{bmatrix}_{\binom{S \backslash X}{2} \times \binom{X}{1}},$$

$$s_2 := \begin{bmatrix} \{1, 2\} \cup \{4\} & \{1, 3\} \cup \{4\} & \{2, 3\} \cup \{4\} \\ \{1, 2\} \cup \{5\} & \{1, 3\} \cup \{5\} & \{2, 3\} \cup \{5\} \\ \{1, 2\} \cup \{6\} & \{1, 3\} \cup \{6\} & \{2, 3\} \cup \{6\} \end{bmatrix}_{\binom{S \backslash X}{1} \times \binom{X}{2}} \quad \text{and}$$

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$$s_3 := [\{1,2,3\}]_{\binom{S\setminus X}{0} \times \binom{X}{3}}.$$

3.3. A partition of $\binom{S}{r}$ by subsets satisfying property $(**)$.

By (3.1.1), we will construct $\mathcal{U}'s$ having property $(**)$ which form a partition of $\binom{S}{r}$. Let $S = \{1,2,...,n\}$, $2 \leq r \leq n-1$ and let $X \in \binom{S}{r}$ be fixed.

3.3.1. Let $0 \leq h \leq r$ and take the $\binom{|S\setminus X|}{r-h} \times \binom{|X|}{h}-$matrix s_h. By (3.1.1.b), we can make $\max\{\binom{n-r}{r-h}, \binom{r}{h}\}$ different sets consisting of the entries of S_h satisfying property $(**)$. Namely, take each set constructing with the entries of each major diagonal of S_h. In this way, we get:

3.3.1.a. $\max\{\binom{n-r}{r-h}, \binom{r}{h}\}$ different sets of cardinality $\min\{\binom{n-r}{r-h}, \binom{r}{h}\}$.

Graphically speaking, for
$\begin{bmatrix} \bullet_1 & \triangleright_2 & \circ_3 \\ *_1 & \bullet_2 & \triangleright_3 \\ \circ_1 & *_2 & \bullet_3 \\ \triangleright_1 & \circ_2 & *_3 \end{bmatrix}_{4\times 3}$
$\longleftrightarrow$
$\begin{bmatrix} \bullet_1 & & \\ *_1 & \bullet_2 & \\ \circ_1 & *_2 & \bullet_3 \\ \triangleright_1 & \circ_2 & *_3 \\ & \triangleright_2 & \circ_3 \\ & & \triangleright_3 \end{bmatrix}$,

we obtain $\{\bullet_1, \bullet_2, \bullet_3\}, \{*_1, *_2, *_3\}, \{\circ_1, \circ_2, \circ_3\}$ and $\{\triangleright_1, \triangleright_2, \triangleright_3\}$.

3.3.1.b. Now, the Partition of $\binom{S}{r}$:

Case $\binom{n-h}{r-h} \geq \binom{r}{h}$. Then $\max\{\binom{n-r}{r-h}, \binom{r}{h}\} = \binom{n-r}{r-h}$.

As we mentioned before, the set of the entries of S_h, $\left\{A_t^{(h)} \cup Z_i^{(h)}\right\}_{1 \leq i \leq \binom{n-r}{r-h},\ 1 \leq t \leq \binom{r}{h}} =$

$\sqcup_{j=1}^{\binom{n-r}{r-h}} \sqcup_{t=1}^{\binom{r}{h}} \left\{A_t^{(h)} \cup Z_{\sigma_j^{(h)}(t)}^{(h)}\right\}$, where for all $j = 1,...,\binom{n-r}{r-h}$, $\sigma_j^{(h)}$ is the set-function $\sigma_j^{(h)} : \{1,2,...,\binom{r}{h}\} \to \{1,2,...,\binom{n-r}{r-h}\}$ given by $\sigma_j^{(h)}(t) = [j+t-1]_{mod \binom{n-r}{r-h}}$.

And by (3.1.1.b), for each $j \in \{1,...,\binom{n-r}{r-h}\}$, $s_h(j) := \sqcup_{t=1}^{\binom{r}{h}} \left\{A_t^{(h)} \cup Z_{\sigma_j^{(h)}(t)}^{(h)}\right\}$ satisfies property $(**)$.

Case $\binom{n-h}{r-h} < \binom{r}{h}$ the proof is similar.

3.3.1.c. In conclusion, for all $0 \leq h \leq r$ and for all $j = 1,...,\max\left\{\binom{n-r}{r-h}, \binom{r}{h}\right\}$,

$$s_h(j) := \sqcup_{t=1}^{\min\{\binom{n-r}{r-k}, \binom{r}{k}\}} \left\{A_t^{(h)} \cup Z_{\sigma_j^{(h)}(t)}^{(h)}\right\} \text{ satisfies property } (**)$$

where $\sigma_j^{(h)}$ is the set-function
$\sigma_j^{(h)} : \left\{1,2,...,\min\left\{\binom{n-r}{r-h}, \binom{r}{h}\right\}\right\} \to \left\{1,2,...,\max\left\{\binom{n-r}{r-h}, \binom{r}{h}\right\}\right\}$ given by
$\sigma_j^{(h)}(t) = [j+t-1]_{mod\ max\{\binom{n-r}{r-h}, \binom{r}{h}\}}$.

And the set of entries of S_h is $\sqcup_{0 \leq h \leq r} \sqcup_{1 \leq j \leq \max\{\binom{n-r}{r-h}, \binom{r}{h}\}} s_h(j)$.

3.3.2. Now, using (3.3.1), we will give bigger subsets of $\binom{S}{r}$ which still have property $(**)$.

Define for $j = 1, ..., \max\limits_{0 \le k \le r,\ k\ \text{odd}} \left\{ \max\left\{ \binom{n-r}{r-k}, \binom{r}{k} \right\} \right\}, \mathcal{U}_j^{(odd)} := \bigsqcup_{0 \le h \le r,\ h\ \text{odd}} s_h(j)$

and for $j = 1, ..., \max\limits_{0 \le k \le r,\ k\ \text{even}} \left\{ \max\left\{ \binom{n-r}{r-k}, \binom{r}{k} \right\} \right\}, \mathcal{U}_j^{(even)} := \bigsqcup_{0 \le h \le r,\ h\ \text{even}} s_h(j)$

where in both cases, for all h, $S_h(j) := \emptyset$ if $j > \max\left\{ \binom{n-r}{r-h}, \binom{r}{h} \right\}$.

3.3.3. Therefore by (3.1.1.c), for each j, $\mathcal{U}_j^{(odd)}$ and $\mathcal{U}_j^{(even)}$ satisfy property
$(**)$, since $\forall h \ne h'$ odd (resp. even) numbers $|h - h'| \ge 2$.
  And $\left\{ \mathcal{U}_j^{(odd)} \right\}_{j=1,...,\max\limits_{0 \le k \le r,\ k\ \text{odd}} \binom{n-r}{r-k}} \bigcup \left\{ \mathcal{U}_j^{(even)} \right\}_{j=1,...,\max\limits_{0 \le k \le r,\ k\ \text{even}} \binom{n-r}{r-k}}$ is a partition of $\binom{S}{r}$.
  It will be useful later, to note that this partition has exactly
  $\gamma := \max_{0 \le h \le r,\ h\ \text{odd}} \{\max\{\binom{n-r}{r-h}, \binom{r}{h}\}\} + \max_{0 \le h \le r,\ h\ \text{even}} \{\max\{\binom{n-r}{r-h}, \binom{r}{h}\}\}$
  elements.∎

3.3.4. Known $\binom{r}{h} = \binom{r}{r-h}$. Thus, $\max\{\binom{n-r}{r-h}, \binom{r}{h}\} = \binom{r}{h} \Leftrightarrow r \ge n - r$.
  $\gamma := \max_{0 \le h \le r,\ h\ \text{odd}} \{\binom{r}{h}\} + \max_{0 \le h \le r,\ h\ \text{even}} \{\binom{r}{h}\} = \binom{r}{[r/2]} + \binom{r}{[(r+1)/2]} \Leftrightarrow$
$r \ge n - r$.

3.3.5. Example. Continue with the example (3.2), we have: $S = \{1, ..., 6\}$
and $r = 3$.
$$s_1 := \begin{bmatrix} \mathbf{(a)}\{1,4,5\} & \mathbf{(c)}\{2,4,5\} & \mathbf{(b)}\{3,4,5\} \\ \mathbf{(b)}\{1,4,6\} & \mathbf{(a)}\{2,4,6\} & \mathbf{(c)}\{3,4,6\} \\ \mathbf{(c)}\{1,5,6\} & \mathbf{(b)}\{2,5,6\} & \mathbf{(a)}\{3,5,6\} \end{bmatrix}.$$

By (3.1.1.b), each of $\{\{1,4,5\},\{2,4,6\},\{3,5,6\}\}, \{\{1,4,6\},\{2,5,6\},\{3,4,5\}\}$
and $\{\{1,5,6\},\{2,4,5\},\{3,4,6\}\}$ satisfies property $(**)$. And by (3.1.1.c),
  $\mathcal{U}_1^{(even)} := \{\{4,5,6\},\{1,2,4\},\{1,3,5\},\{2,3,6\}\},$
    $\mathcal{U}_2^{(even)} := \{\{1,2,5\},\{1,3,6\},\{2,3,4\}\},$
    $\mathcal{U}_3^{(even)} := \{\{1,2,6\},\{1,3,4\},\{2,3,5\}\},$
    $\mathcal{U}_1^{(odd)} := \{\{1,4,5\},\{2,4,6\},\{3,5,6\}\},$
    $\mathcal{U}_2^{(odd)} := \{\{1,4,6\},\{2,5,6\},\{3,4,5\}\}$ and
    $\mathcal{U}_3^{(odd)} := \{\{1,5,6\},\{2,4,5\},\{3,4,6\}\}.$

That is, each $\mathcal{U}_i$ satisfies property $(**)$ and $\binom{S}{3} = \bigsqcup_{i=1}^3 \mathcal{U}_i^{(odd)} \sqcup \bigsqcup_{i=1}^3 \mathcal{U}_i^{(even)}$.

4 $\left|\mathbf{Matroid}_{n,r}\right| \le 2\binom{r}{r/2} \left|\mathbf{Sparse}_{n,r}\right|$ if $r \ge \frac{n}{2}$.

Recall that $\text{Matroid}_{n,r}$ (resp., $\text{Sparse}_{n,r}$) is the set of all matroids (resp.
sparse-paving matroids) on a set S of cardinality n and rank r. For other bounds
of the cardinalities of these sets, see [13], [8], [10], **??**, [1], [**?**].

Let M be a matroid on S of rank r. Then the $r-$subsets of S, $\binom{S}{r} = \mathcal{B} \sqcup \mathcal{D}_r \sqcup \mathcal{C}_r$ is the disjoint union of its set of basis, $\mathcal{B}$, its set of r-circuits, $\mathcal{C}_r$ and the dependent r-subsets which are no circuits, $\mathcal{D}_r$.

Now following the notation of (3.3), all the sets $(\mathcal{C}_r \cup \mathcal{D}_r) \cap \mathcal{U}_j^{(odd)}$ and $(\mathcal{C}_r \cup \mathcal{D}_r) \cap \mathcal{U}_j^{(even)}$ satisfy property $(**)$. That is, each of these sets define sparse-paving matroids, see (1.3).

4.1. Theorem: *The follow set-function Ψ_r is injective, where*

$\alpha := \max_{0 \leq h \leq r,\ h\ odd}\{\max\{\binom{n-r}{r-h}, \binom{r}{h}\}\}$ *and*

$\beta := \max_{0 \leq h \leq r,\ h\ even}\{\max\{\binom{n-r}{r-h}, \binom{r}{h}\}\}$.

$\Psi_r : Matroid_{n,r} \to \sqcup_{j=1}^{\alpha} Sparse_{n,r} \times \{j\} \sqcup \sqcup_{j=\alpha+1}^{\alpha+\beta} Sparse_{n,r} \times \{j\}$ *such that*
$\Psi_r(M) = \sqcup_{j=1}^{\alpha+\beta}\left(M_j^{(r)}, j\right)$ *where*

$M_j^{(r)}$ *is the sparse-paving matroid with set of $r-$circuits $(\mathcal{C}_r \cup \mathcal{D}_r) \cap U_j^{(odd)}$ if $1 \leq j \leq \alpha$ and*

$M_j^{(r)}$ *is the sparse-paving matroid with set of $r-$circuits $(\mathcal{C}_r \cup \mathcal{D}_r) \cap U_j^{(even)}$ if $\alpha + 1 \leq j \leq \alpha + \beta$.*

Proof. By (1.3), Ψ_r is a well defined set-function, and since any matroid is defined by its sets of basis, Ψ_r is injective.■

4.2. Another version of (4.1) is the following, whose idea is to recognize in any matroid its $r-$circuits from its dependent $r-$subsets which are not circuits.

Theorem: *The follow set-function $\overline{\Psi}_r$ is injective, where*

$\alpha := \max_{0 \leq h \leq r,\ h\ odd}\{\max\{\binom{n-r}{r-h}, \binom{r}{h}\}\}$ *and*

$\beta = \max_{0 \leq h \leq r,\ h\ even}\{\max\{\binom{n-r}{r-h}, \binom{r}{h}\}\}$.

$\overline{\Psi}_r : Matroid_{n,r} \to \sqcup_{j=1}^{\alpha} Sparse_{n,r} \times \{(c,j)\} \sqcup \sqcup_{j=\alpha+1}^{\alpha+\beta} Sparse_{n,r} \times \{(c,j)\} \sqcup \sqcup_{j=1}^{\alpha} Sparse_{n,r} \times \{(d,j)\} \sqcup \sqcup_{j=\alpha+1}^{\alpha+\beta} Sparse_{n,r} \times \{(d,j)\}$

such that $\overline{\Psi}_r(M) = \sqcup_{j=1}^{\alpha+\beta}\left(M_{c,j}^{(r)}, (c,j)\right) \sqcup \sqcup_{j=1}^{\alpha+\beta}\left(M_{d,j}^{(r)}, (d,j)\right)$ *where*

$M_{(c,j)}^{(r)}$ *is the sparse-paving matroid with set of $r-$circuits $\mathcal{C}_r \cap \mathcal{U}_j^{(odd)}$ if $1 \leq j \leq \alpha$,*

$M_{(d,j)}^{(r)}$ *is the sparse-paving matroid with set of $r-$circuits $\mathcal{D}_r \cap \mathcal{U}_j^{(odd)}$ if $\alpha + 1 \leq j \leq \alpha + \beta$,*

$M_{(c,j)}^{(r)}$ *is the sparse-paving matroid with set of $r-$circuits $C_r \cap U_j^{(even)}$ if $1 \leq j \leq \alpha$ and*

$M_{(d,j)}^{(r)}$ *is the sparse-paving matroid with set of $r-$circuits $D_r \cap U_j^{(even)}$ if $\alpha + 1 \leq j \leq \alpha + \beta$.*■

4.3. With the same proof of (4.1), we have the next

Proposition. *The set-function Γ_r is injective. Let denote by $M_\mathcal{C}$ the sparse-paving matroid with C its $r-$circuits.*

$$\Gamma_r : \text{Matroid}_{n,r} \to \quad \sqcup_{j=1}^{\alpha} \left\{ M_{\mathcal{C}} \in \text{Sparse}_{n,r}; \mathcal{C} \subseteq \mathcal{U}_j^{(odd)} \right\} \times \{j\} \sqcup$$
$$\sqcup_{j=\alpha+1}^{\alpha+\beta} \left\{ M_{\mathcal{C}} \in \text{Sparse}_{n,r}; \ \mathcal{C} \subseteq \mathcal{U}_j^{(even)} \right\} \times \{j\}$$

such that $\Gamma_r(M) = \sqcup_{j=1}^{\alpha+\beta} \left(M_{\mathcal{C}_j}, j \right)$ *where* $C_j := (\mathcal{C}_r \cup \mathcal{D}_r) \cap U_j^{(odd)}$ *if* $1 \leq j \leq \alpha$
and $C_j := (\mathcal{C}_r \cup \mathcal{D}_r) \cap U_j^{(even)}$ *if* $\alpha + 1 \leq j \leq \alpha + \beta.$∎

4.4. Corollary: *With the same notations of the section and (2.4). Let*
$\gamma = \max_{0 \leq h \leq r, \ h \ odd} \{\max\{\binom{n-r}{r-h}, \binom{r}{h}\}\} + \max_{0 \leq h \leq r, \ h \ even} \{\max\{\binom{n-r}{r-h}, \binom{r}{h}\}\}$. *Then*

a) $2^{\left[\frac{1}{r(n-r)+1} \binom{n}{r} \right]} \leq |\text{Sparse}_{n,r}| \leq |\text{Matroid}_{n,r}| \leq \gamma \, |\text{Sparse}_{n,r}|$.

b) $|\text{Matroid}_{n,r}| \leq 2^{\gamma \left[\frac{1}{n-r} \binom{n}{r+1} \right]}$.

Proof. (b) By (4.3) and (2.2), for $\mathcal{U}$ satisfying property $(**)$, $|\mathcal{U}| \leq \frac{1}{n-r} \binom{n}{r+1}$.
Then $\left| \left\{ M_{\mathcal{C}} \in \text{Sparse}_{n,r}; \mathcal{C} \subseteq \mathcal{U} \right\} \right| = 2^{|\mathcal{U}|} \leq 2^{\frac{1}{n-r} \binom{n}{r+1}}$ and $|\text{Matroid}_{n,r}| \leq 2^{\gamma \left[\frac{1}{n-r} \binom{n}{r+1} \right]}$.∎

4.5. Observations for γ.
$\gamma := \alpha + \beta = \max_{0 \leq h \leq r, \ h \ odd} \{\max\{\binom{n-r}{r-h}, \binom{r}{h}\}\} + \max_{0 \leq h \leq r, \ h \ even} \{\max\{\binom{n-r}{r-h}, \binom{r}{h}\}\}$.

i) $\gamma = \binom{r}{[r/2]} + \binom{r}{[(r+1)/2]} = 2\binom{r}{[r/2]} \Leftrightarrow n-1 \geq r \geq n-r \Leftrightarrow n-1 \geq r \geq \frac{n}{2}$.
And it is known, $\frac{2^{r+1}}{r+1} \leq \gamma = 2\binom{r}{[r/2]} \underset{r \to \infty}{\to} \frac{2^r}{\sqrt{\frac{\pi r}{2}}} < 2^r$.

ii) In case, $r \leq \frac{n}{2}$, we have two cases:

ii.1) If $2 \leq r \leq \frac{n-r}{2} (\Leftrightarrow 2 \leq r \leq \frac{n}{3})$, $\gamma = \binom{n-r}{r-1} + \binom{n-r}{r} = \binom{n-r+1}{r} \leq$
$\binom{n-r+1}{[(n-r)/2]} \leq \binom{n-r+1}{[(n-r+1)/2]} \underset{r \to \infty}{<} 2^{n-r+1} < 2^{\frac{2n+3}{3}}$

ii.2) If $\frac{n-r}{2} \leq r \leq \frac{n}{2} (\Leftrightarrow \frac{n}{3} \leq r \leq \frac{n}{2})$, $\gamma = 2\binom{n-r}{[(n-r)/2]}$. And $\frac{2^{n-r+1}}{n-r+1} \leq \gamma = 2\binom{n-r}{[(n-r)/2]} \underset{r \to \infty}{\to} \frac{2^{n-r}}{\sqrt{\frac{\pi(n-r)}{2}}} < 2^{n-r} < 2^{\frac{n}{2}}$

iii) Therefore, $\gamma < \begin{cases} 2^r & \text{if } \frac{n}{2} \leq r \leq n-1 \\ 2^{n-r+1} & \text{if } 2 \leq r \leq \frac{n}{3} \\ 2^{n-r} & \text{if } \frac{n}{3} \leq r \leq \frac{n}{2} \end{cases}$ and

$\log_2 \gamma < \begin{cases} r & \text{if } \frac{n}{2} \leq r \leq n-1 \\ n-r+1 & \text{if } 2 \leq r \leq \frac{n}{3} \\ n-r & \text{if } \frac{n}{3} \leq r \leq \frac{n}{2} \end{cases}$

iv) $\frac{1}{r(n-r)+1} \binom{n}{r} \geq \begin{cases} \frac{2}{n^2-n+2} \binom{n}{n/2} & \text{if } \frac{n}{2} \leq r \leq n-1 \\ \frac{3}{n^2-2n+3} \binom{n}{2} & \text{if } 2 \leq r \leq \frac{n}{3} \\ \frac{3}{n^2+3} \binom{n}{n/3} & \text{if } \frac{n}{3} \leq r \leq \frac{n}{2} \end{cases}$.
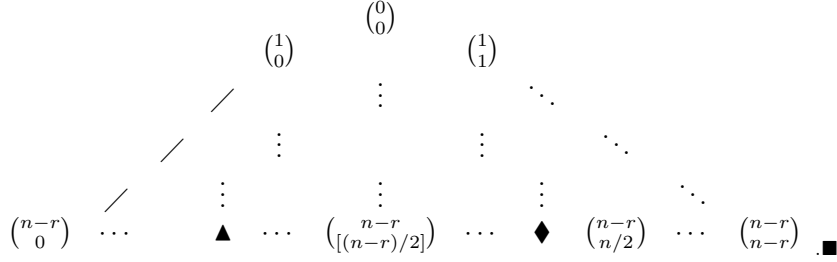
The proof is straight forward.

Proof. (ii) Looking at the Pascal's triangle and $r \leq \frac{n}{2}$ we have two cases:
(ii.1) If $r \leq \frac{n-r}{2}$ then $\max_{0 \leq h \leq r} \{\max\{\binom{n-r}{r-h}, \binom{r}{h}\}\} = \binom{n-r}{r}$ in ▲−region.
(ii.2) If $\frac{n-r}{2} \leq r \leq \frac{n}{2}$ then $\max_{0 \leq h \leq r} \{\max\{\binom{n-r}{r-h}, \binom{r}{h}\}\} = \binom{n-r}{[(n-r)/2]}$ in
♦−region.

Pascal's triangle:

$$\begin{array}{c}
\binom{0}{0}\\
\binom{1}{0} \qquad\qquad \binom{1}{1}\\
\diagup \qquad \vdots \qquad\qquad \ddots\\
\diagup \qquad \vdots \qquad \vdots \qquad \ddots\\
\diagup \qquad \vdots \qquad \vdots \qquad \vdots \qquad \ddots\\
\binom{n-r}{0} \;\cdots\; \blacktriangle \;\cdots\; \binom{n-r}{[(n-r)/2]} \;\cdots\; \blacklozenge \;\; \binom{n-r}{n/2} \;\cdots\; \binom{n-r}{n-r}\,.\blacksquare
\end{array}$$

4.6. Corollary. $\lim_{n\to\infty} \frac{\log_2|\mathrm{Matroid}_{n,r}|}{\log_2|\mathrm{Sparse}_{n,r}|} = 1$ and $\lim_{n\to\infty} \frac{\log_2|\mathrm{Matroid}_n|}{\log_2|\mathrm{Sparse}_n|} = 1$ where (Sparse_n) $\mathrm{Matroid}_n$ is the set of the (sparse-paving) matroids over S, $|S| = n$.

Proof. By (4.5),

$$\begin{aligned}
1 \le \lim_{n\to\infty} \frac{\log_2|\mathrm{Matroid}_{n,r}|}{\log_2|\mathrm{Sparse}_{n,r}|} &\le \lim_{n\to\infty} \left(\frac{\log_2|\mathrm{Sparse}_{n,r}|}{\log_2|\mathrm{Sparse}_{n,r}|} + \frac{\log_2 \gamma}{\log_2|\mathrm{Sparse}_{n,r}|} \right)\\
&\le 1 + \lim_{n\to\infty} \frac{\log_2 \gamma}{2^{\left[\frac{1}{r(n-r)+1}\binom{n}{r}\right]}}
\end{aligned}.$$

On the other hand, by the duality $M \leftrightarrow M^*$, we have $\left|\mathrm{Sparse}_{n,r}\right| = \left|\mathrm{Sparse}_{n,n-r}\right|$. Then without loss of generality we can assume $n - 1 \ge r \ge \frac{n}{2}$.
Then by (4.4), $2^{\left[\frac{1}{r(n-r)+1}\binom{n}{r}\right]} \ge 2^{\frac{2}{n^2-n+2}\binom{n}{n/2}} > 2^{\frac{4^n}{n^2-n+2}}$.
Thus, $0 \le \lim_{n\to\infty} \frac{\log_2 \gamma}{2^{\left[\frac{1}{r(n-r)+1}\binom{n}{r}\right]}} \le \lim_{n\to\infty} \frac{\max\{r,n-r+1\}}{2^{\frac{4^n}{n^2-n+2}}} = \lim_{n\to\infty} \frac{n+2}{2^{\frac{4^n}{n^2-n+2}+1}} = 0$.

Therefore, $\lim_{n\to\infty} \frac{\log_2|\mathrm{Matroid}_{n,r}|}{\log_2|\mathrm{Sparse}_{n,r}|} = 1$.$\blacksquare$

References

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